Multivariate Gram-Charlier Densities*

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Abstract

This paper introduces a new family of multivariate distributions based on Gram-Charlier and Edgeworth expansions. This family encompasses many of the univariate Edgeworth-Sargan (hereafter ES) or Gram-Charlier densities as marginal distributions of the different formulations. Within this family, we focus on the specifications that guarantee positivity so obtaining a well-defined multivariate density. We compare different "positive" multivariate distributions of the family with both the multivariate ES and the Normal in an in- and out-sample framework for financial returns data. Our results show that the proposed specifications provide a quite reasonably good performance being so of interest for applications involving the modelling and forecasting of heavy-tailed distributions.

Key words: Multivariate distributions; Gram-Charlier and Edgeworth-Sargan densities; MGARCH models; Financial data.

JEL classification: C16, G1.

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1 Introduction

The Gram-Charlier and the Edgeworth expansions were established at the end of the 19th century and the beginning of the 20th century by Edgeworth (1896, 1907) and Charlier (1905). Since then, these expansions have been used in many fields from mathematics or statistics to physics, but it was Sargan in the 70s who brought these expansions into econometrics — Sargan (1975, 1976). More recently, the literature on this topic has increased from both theoretical studies — e.g. Nishiyama and Robinson (2000), Velasco and Robinson (2001) and Nabeya (2001) — and applications in finance to fit the heavy-tailed distribution of high-frequency asset returns — Corrado and Su (1996), Mauleón and Perote (2000) and Verhoeven and McAleer (2004), among others.

The latter articles provide evidence of the good performance of these distributions, besides of, the widely known nonpositivity curse that they present when they need to be truncated in practical applications, as firstly highlighted by Barton and Dennis (1952) and Draper and Tierny (1972). Different solutions has been proposed to this problem in univariate contexts by authors such as Gallant and Nychka (1987), Gallant and Tauchen (1989), Jondeau and Rockinger (2001), Ñíguez and Perote (2004) and Leon et al. (2005).

On the other hand, the multivariate context may be of greater interest to explore the possible gains in in-sample fit and forecasting when accounting for the joint distribution of correlated variables. For such purposes different approaches have been proposed, including: Multivariate Skewed Normal (Azzalini and Dalla Valle, 1996), Multivariate Student's t (Kotz and Nadarajah, 2004), Multivariate Weibull (Malevergne and Sornette, 2004), Kotz-type distributions (Olcay, 2005) and copula methods (e.g., Xiaohong et al., 2006) and multivariate GARCH models (Bauwens et al., 2005). Nevertheless, to the knowledge of the authors, the Gram-Charlier and Edgeworth expansions have scarcely been studied in a multivariate context.[1] Recently, Perote (2004) has proposed a closed form for a multivariate density

[1]See, e.g., Henery (1981) for a particular case of uncorrelated variables applied to model disease

based on the so-called Edgeworth-Sargan distribution (ES henceforth). The resulting density, called Multivariate ES (MES henceforth), showed a good performance to fit the multivariate distribution of financial returns.[2] But, as the ES, the MES distribution also presents the problem of not being positive for all values of the parameters in the parametric space.[3] The main aim of this article is to shed some light on this issue by providing a family of positive multivariate Gram-Charlier distributions that generalise the univariate positive Gram-Charlier distributions proposed in the literature.

The remainder of the article is structured as follows. Section 2 deals with the definitions of the multivariate Gram-Charlier families of densities. Section 3 tests the performance of the proposed densities through an empirical application for financial data, and Section 4 presents the main conclusions and suggests possible lines for further research.

2 Multivariate Gram-Charlier Densities

In this section we propose a general multivariate family of distributions based on the semi-nonparametric (SNP henceforth) density approach derived from the Edgeworth or Gram-Charlier expansions. This family encompasses most of the univariate ES used in the literature to model and forecast high-frequency financial returns for risk management purposes. The "standardised" Multivariate Gram-Charlier (MGC hereafter) family of densities is defined in terms of the "standardised" multivariate Normal (MN hereafter) density, $G(\cdot)$, (i.e. with unitary variance for all its marginal densities, $g(\cdot)$, and correlation coefficients denoted by ρ_{ij} $i, j = 1, \ldots, n$), and the so-called Hermite polynomials, $H_s(\cdot)$, as given in the definition below.[4]

transmission.

[2] See also Perote and Del Brio (2003) for applications of the MES distribution to estimate Value-at-Risk (VaR henceforth) measures.

[3] Note that we refer to the ES distribution as an already truncated Edgeworth expansion.

[4] Note that although we define the "standardised" MGC densities in terms of Gaussian densities with unitary variance, the resulting distributions do not have unitary variance, since variances, as the rest of the

Definition 1 *A random vector* $X = (x_1, x_2, \ldots, x_n)' \in \mathbb{R}^n$ *belongs to the MGC family of distributions if it is distributed according to*

$$F(\mathbf{X}) = \frac{1}{n+1} G(\mathbf{X}) + \frac{1}{n+1} \left\{ \prod_{i=1}^{n} g\left(x_i\right) \right\} \left\{ \sum_{i=1}^{n} \frac{1}{c_i} \mathbf{h}_i\left(x_i\right)' \mathbf{A}_i \mathbf{h}_i\left(x_i\right) \right\}, \tag{1}$$

where A_i *is a matrix of order* $(q+1)$,[5] $h_i\left(x_i\right) = (1, H_1(x_i), H_2(x_i), \cdots, H_q(x_i))' \in \mathbb{R}^{q+1}$, $H_s(\cdot)$ *stands for the* $s-th$ *order Hermite polynomial described in equation (2),*

$$H_s(x_i) = \begin{cases} \sum_{i=0}^{s/2} (-1)^i x_i^{s-2i} \frac{s!}{2^i i! (s-2i)!} & \forall s \text{ is even} \\ \sum_{i=0}^{(s-1)/2} (-1)^i x_i^{s-2i} \frac{s!}{2^i i! (s-2i)!} & \text{otherwise,} \end{cases} \tag{2}$$

and c_i *is the constant such that*

$$c_i = \int g\left(x_i\right) \mathbf{h}_i\left(x_i\right)' A_i \mathbf{h}_i\left(x_i\right) dx_i. \tag{3}$$

Note that the MGC family of functions straightforwardly integrates up to one and represents density functions providing that $\mathbf{A}_i$ is a positive definite matrix. An interesting and simple case arises when $\mathbf{A}_i$ admits the following decomposition, $\mathbf{A}_i = \mathbf{d}_i \mathbf{d}_i'$, with $\mathbf{d}_i = (1, d_{i1}, \ldots, d_{iq})' \in \mathbb{R}^{q+1}$ containing the density parameters of the $i-th$ density dimension. For this particular case, a positive version of the MGC density, hereafter named as MGCI, can be defined as in equation (4) below,

$$F_I(\mathbf{X}) = \frac{1}{n+1} G(\mathbf{X}) + \frac{1}{n+1} \left\{ \prod_{i=1}^{n} g\left(x_i\right) \right\} \left\{ \sum_{i=1}^{n} \frac{1}{c_i} \left[1 + \sum_{s=1}^{q} d_{is} H_s\left(x_i\right) \right]^2 \right\} \tag{4}$$

For this density, and based on the well-known orthogonality properties given in equations (5) and (6),[6]

$$\int H_s(x_i) H_j(x_i) g(x_i) dx_i = 0 \quad \forall s \neq j \tag{5}$$

$$\int H_s(x_i) H_j(x_i) g(x_i) dx_i = s! \quad \forall s = j, \tag{6}$$

it can be straightforwardly shown that,

density moments, depend on the whole set of density parameters.

[5] Note that without loss of generality we have considered that for all dimensions the Gram-Charlier (type A) expansions are truncated at the same order q.

[6] See Kendall and Stuart (1977) for further details about the Hermite polynomials properties.

(i) The constant that weights the squared sum of Hermite polynomials for every variable x_i (see Proof 1 in the Appendix) is

$$c_i = \int g(x_i) \left[1 + \sum_{s=1}^{q} d_{is} H_s(x_i) \right]^2 dx_i = 1 + \sum_{s=1}^{q} d_{is}^2 s!. \tag{7}$$

(ii) The density integrates up to one, (see Proof 2 in the Appendix).

(iii) The marginal density for variable x_i is a mixture of a univariate Normal and a univariate Gram-Charlier density of the type recently analysed in Leon et al. (2005, 2006), as shown in equation (8) (see Proof 2 in the Appendix).

$$f_I(x_i) = \frac{n}{n+1} g(x_i) + \frac{1}{(n+1)} \frac{1}{c_i} g(x_i) \left[1 + \sum_{s=1}^{q} d_{is} H_s(x_i) \right]^2 . \tag{8}$$

Note that the MCGI distribution overcomes the aforementioned nonpositivity problem that may arise in applying the MES density in Perote (2004), equation (9).[7]

$$F_{ES}(\mathbf{X}) = G(\mathbf{X}) + \left\{ \prod_{i=1}^{n} g(x_i) \right\} \left\{ \sum_{i=1}^{n} \sum_{s=1}^{q} d_{is} H_s(x_i) \right\}, \tag{9}$$

On the other hand, the MCGI density may also be interpreted as a multivariate generalisation of Gallant and Nychka's methodology.[8]

An interesting feature of the MCG family of densities proposed in this paper is its general formulation since it may encompass many different "positive" multivariate densities. For instance, another possible alternative is the density defined in equation (10) below, which we denote as MGCII,

$$F_{II}(\mathbf{X}) = \frac{1}{n+1} G(\mathbf{X}) + \frac{1}{n+1} \left\{ \prod_{i=1}^{n} g(x_i) \right\} \left\{ \sum_{i=1}^{n} \frac{1}{c_i} \left[1 + \sum_{s=1}^{q} d_{is}^2 H_s(x_i)^2 \right] \right\} \tag{10}$$

[7]Note that for the maximum likelihood estimates the MES is necessarily positive and thus this density can be estimated in many applications by choosing accurate initial values, based on the estimates for its marginal densities that are distributed as the univariate ES in Mauleon and Perote (2000).

[8]See Fenton and Gallant (1996) for a detailed analysis of the properties of the SNP densities defined when truncating Edgeworth and Gram-Charlier expansions.

For this density the scaling constants, c_i, are also those in equation (7), but its marginals, equation (11), are mixtures of a univariate normal and the univariate positive ES (PES) of Ñíguez and Perote (2004).

$$f_{II}(x_i) = \frac{n}{n+1}g(x_i) + \frac{1}{(n+1)}\frac{1}{c_i}g(x_i)\left[1+\sum_{s=1}^{q} d_{is}^2 H_s(x_i)^2\right]. \tag{11}$$

Furthermore, the MGCII cumulative distribution functions (cdf hereafter) can be easily worked out as shown in equation (12) — see Proof 4 in the Appendix, and consequently, they can be used easily for risk management purposes, either for modelling and forecasting credit risk, portfolios VaR or shortfall probabilities.

$$\begin{aligned} & \Pr\left[x_1 \leq a_1, \cdots, x_n \leq a_n\right] \\ = & \frac{1}{n+1}\int_{-\infty}^{a_1}\cdots\int_{-\infty}^{a_n} G(\mathbf{X})dx_1\cdots dx_n + \frac{1}{n+1}\prod_{\substack{j=1\\ j\neq i}}^{n-1}\int_{-\infty}^{a_j} g(x_j)\,dx_j \\ & \times\sum_{i=1}^{n}\left[\int_{-\infty}^{a_i} g(x_i)\,dx_i - \frac{g(a_i)}{c_i}\sum_{s=1}^{q} d_{is}^2\sum_{k=0}^{s-1}\frac{s!}{(s-k)!}H_{s-k}(a_i)H_{s-k-1}(a_i)\right] \end{aligned} \tag{12}$$

The MGC densities straightforwardly admit the specification of GARCH-type processes (Engle (1982) and Bollerslev (1986)) to explain the dynamics of their conditional moments. Particularly, the conditional variances, k_{it}^2, are introduced by considering transformations of the type $.\mathbf{Z}_t = \mathbf{\Lambda X}$ where $\Lambda = diag(k_{1t}, k_{2t}, \cdots, k_{nt})$.[9] In the next section we test the performance of the bivariate versions of these new densities in comparison with the MES and the MN through an empirical exercise for stock returns.

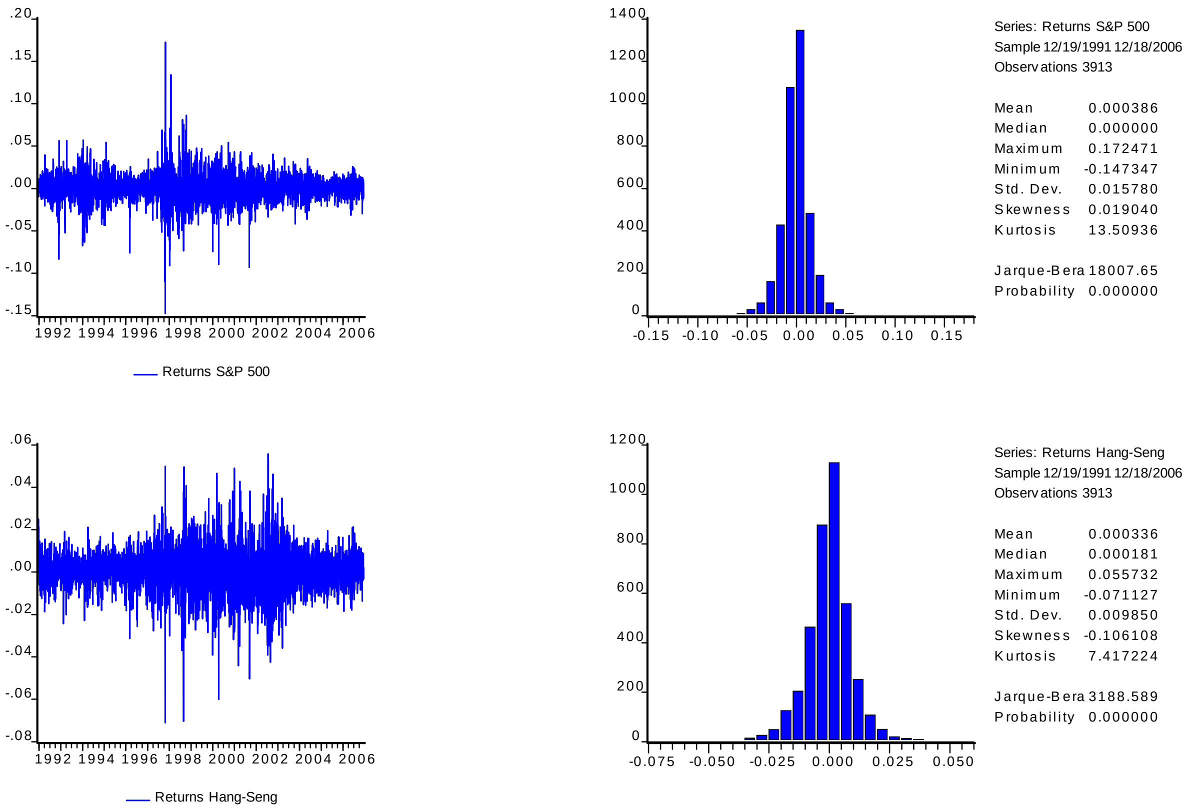
[9] Note that this model is a case of the constant conditional correlation (CCC) model of Bollerslev, where the constant correlations coefficients are those ρ_{ij} included in the "standardised" MN of the MGC densities.

3 Empirical application

3.1 Data, estimation and in-sample analysis

The data used are daily returns of S&P500 and the Hang-Seng indices of the New York and Hong Kong Stock Exchange, respectively, $\mathbf{r}_t = (r_{1t}, r_{2t})$, over the period December 19, 1991 to December 19, 2006 for a total of $T = 3,913$ observations, obtained from Datastream. A plot and descriptive statistics of r_t is presented in Figure 1.

Figure 1. S&P 500 and Hang-Seng indices daily returns. Sample: 19/12/1991 - 06/06/2005 (observations 3,513). Out-of-sample: 07/06/1995 - 19/12/2006 (observations 400).



Let the conditional distribution of $\mathbf{r}_t$, be either MN, MGCI, or MCGII, with conditional mean and covariance matrix following AR(1) and CCC (Bollerslev, 1990) processes, respectively.

The estimation procedure is carried out in two steps using an in-sample window of $S = 3,513$ observations. Firstly, the AR(1) process is estimated by ordinary least squares and, secondly, covariance matrix coefficients and density function parameters are estimated by (quasi)-maximum likelihood ((Q)ML) using the AR(1) residuals from the first step. Robust QML covariance estimators are calculated by means of Bollerslev and Wooldridge (1992) formula. The Hermite polynomials expansions were truncated in the $8th$ term according to accuracy criteria. Moreover, the odd parameters of the expansions were constrained to zero because they were found not to be statistically significant.[10]

Tables 1 and 2 display the estimates for the parameters of the bivariate densities when variances are assumed to be either constant or follow a GARCH(1,1), respectively.[11] Under the former assumption the scale parameter of variable i is denoted by k_i and under the latter α_{ij} $\forall j = 0, 1, 2$ and $\forall i = 1, 2$ stand for the parameters of the GARCH(1,1) model used for the conditional variances. Moreover, ϕ_{ij} $\forall j = 0, 1$ are the intercept and the slope, respectively, of the AR(1) process used to capture the conditional mean of every variable i.[12] The rest of the parameters displayed in both tables follow the same notation used throughout the paper.

[10] There is evidence in favour of the existence of skewness in financial returns when conditional skewness is considered, e.g. Harvey and Siddique (1999). Nevertheless, for all the analysed cases the MGC densities where found to be unconditionally symmetric. Moreover, the introduction of conditional patterns for moments other than the variance (e.g. for skewness or kurtosis) when positivity constraints are imposed in Gram-Charlier expansions is still an open question.

[11] It is worth mentioning that the estimation of the joint distribution of a larger number of assets is not an extremely difficult task since the number of parameters only increases proportionally to the number of dimensions. Therefore, the more complex structures can be estimated by implementing algorithms that use the estimates for the marginal densities as initial values for the multivariate distribution.

[12] Note that the AR(1) coefficients missing in Table 2 are the same as those in Table 1.

Table 1: Multivariate densities with constant variance and covariance matrix.

	MN	MES	MGCI	MGCII
$\widehat{\phi}_{10}$	.34x10^{-3}	.34x10^{-3}	.34x10^{-3}	.34x10^{-3}
	(2.14)	(2.14)	(2.14)	(2.14)
$\widehat{\phi}_{11}$	-.13x10^{-1}	-.13x10^{-1}	-.13x10^{-1}	-.13x10^{-1}
	(-0.83)*	(-0.83)*	(-0.83)*	(-0.83)*
$\widehat{k}_1$	.98x10^{-2}	.91x10^{-2}	.14x10^{-1}	.78x10^{-2}
	(88.03)	(54.55)	(75.59)	(52.87)
$\widehat{d}_{12}$		.13	-.77	-.40
		(3.36)	(-11.95)	(-8.49)
$\widehat{d}_{14}$		.21	.74x10^{-1}	.57x10^{-1}
		(8.97)	(7.02)	(4.42)
$\widehat{d}_{16}$		.37x10^{-1}	-.91x10^{-2}	-.76x10^{-2}
		(6.16)	(-5.27)	(-3.49)
$\widehat{d}_{18}$		.45x10^{-1}	.22x10^{-2}	-.12x10^{-2}
		(8.42)	(9.83)	(-6.23)
$\widehat{\rho}_{12}$	.11	.58x10^{-1}	.22	.36
	(7.18)	(5.76)	(7.32)	(9.15)
$\widehat{\phi}_{20}$	.38x10^{-3}	.38x10^{-3}	.38x10^{-3}	.38x10^{-3}
	(1.49)*	(1.49)*	(1.49)*	(1.49)*
$\widehat{\phi}_{21}$	.27x10^{-1}	.27x10^{-1}	.27x10^{-1}	.27x10^{-1}
	(1.68)*	(1.68)*	(1.68)*	(1.68)*
$\widehat{k}_2$	.15x10^{-1}	.17x10^{-1}	.96x10^{-2}	.13x10^{-1}
	(88.03)	(38.86)	(55.82)	(64.61)
$\widehat{d}_{22}$		-.17	-.33	-.24
		(-4.58)	(-6.72)	(-5.42)
$\widehat{d}_{24}$		.17	.77x10^{-1}	.63x10^{-1}
		(8.69)	(5.91)	(8.17)
$\widehat{d}_{26}$		.23x10^{-1}	-.23x10^{-1}	.53x10^{-9}
		(4.19)	(10.47)	(.14x10^{-6})*
$\widehat{d}_{28}$		.41x10^{-2}	.10x10^{-2}	-.11x10^{-2}
		(8.42)	(9.04)	(-8.15)
lnL	25209.0	25822.3	24519.6	24329,7
BIC	-25196.6	-25776.8	-24474.1	-24284.2

Bivariate density for S&P500 (variable 1) and Hang-Seng (variable 2) indices. t-ratios in parentheses. * Non-significant at 5% confidence level.

Table 2: Multivariate densities with conditional variance and covariance matrix.

	MN	MES	MGCI	MGCII
$\widehat{\alpha}_{10}$	$.58\text{x}10^{-6}$	$.49\text{x}10^{-6}$	$.16\text{x}10^{-5}$	$.37\text{x}10^{-6}$
	(3.82)	(3.06)	(3.83)	(3.45)
$\widehat{\alpha}_{11}$	.94	.95	.94	.94
	(102.00)	(95.69)	(102.12)	(96.49)
$\widehat{\alpha}_{12}$	$.64\text{x}10^{-1}$	$.64\text{x}10^{-1}$	$.56\text{x}10^{-1}$	$.48\text{x}10^{-1}$
	(8.90)	(7.04)	(8.68)	(7.95)
$\widehat{d}_{12}$		$-.39\text{x}10^{-1}$	-.97	-.39
		(-.67)*	(-7.88)	(-7.41)
$\widehat{d}_{14}$		$.71\text{x}10^{-1}$	.11	$.13\text{x}10^{-1}$
		(4.95)	(6.91)	(.49)*
$\widehat{d}_{16}$		$.77\text{x}10^{-2}$	$-.15\text{x}10^{-1}$	$-.92\text{x}10^{-15}$
		(1.97)	(-5.33)	(0.00)*
$\widehat{d}_{18}$		$.18\text{x}10^{-2}$	$.18\text{x}10^{-2}$	$.77\text{x}10^{-3}$
		(4.85)	(6.51)	(5.33)
$\widehat{\rho}_{12}$	.11	$.97\text{x}10^{-1}$	.21	.24
	(7.05)	(7.04)	(7.04)	(6.76)
$\widehat{\alpha}_{20}$	$.17\text{x}10^{-5}$	$.14\text{x}10^{-5}$	$.40\text{x}10^{-6}$	$.95\text{x}10^{-6}$
	(3.84)	(3.22)	(3.09)	(3.28)
$\widehat{\alpha}_{21}$	.93	.94	.95	.94
	(104.75)	(79.01)	(95.10)	(83.39)
$\widehat{\alpha}_{22}$	$.69\text{x}10^{-1}$	$.54\text{x}10^{-1}$	$.57\text{x}10^{-1}$	$.38\text{x}10^{-1}$
	(9.78)	(7.21)	(7.13)	(7.48)
$\widehat{d}_{22}$		.14	$-.33\text{x}10^{-1}$	-.73
		(2.73)	(-.63)*	(-7.61)
$\widehat{d}_{24}$		.15	$.48\text{x}10^{-1}$	$.36\text{x}10^{-1}$
		(7.12)	(6.65)	(1.11)*
$\widehat{d}_{26}$		$.27\text{x}10^{-1}$	$-.35\text{x}10^{-2}$	$.11\text{x}10^{-1}$
		(4.46)	(-3.67)	(4.54)
$\widehat{d}_{28}$		$.38\text{x}10^{-2}$	$.68\text{x}10^{-3}$	$.96\text{x}10^{-3}$
		(6.51)	(6.02)	(3.48)
lnL	26426.1	26623.0	25318.8	25266,3
BIC	-26397.1	-26561.0	-25256.9	-25204.3

Bivariate density for S&P500 (variable 1) and Hang-Seng (variable 2) indices. t-ratios in parentheses. * Non-significant at 5% confidence level.

Both tables include the corresponding t-statistic for each parameter (in parentheses), the log-likelihood value ($\ln L$) and the Schwarz Bayesian Information Criterion (BIC), defined as $BIC = -\ln L + p\ln(T)/2$ where p stands for the number of the parameters of the model. Orientatively, and according to these accuracy criteria the MES seem to outperform the other distributions and the performance of the distributions that incorporate time-varying variances are also superior than their unconditional counterparts. Regarding to the parameter estimates most of them are significant at the 5 per cent confidence level, confirming the fact that the proposed models seem to be accurate fits of the density underlying the S&P500 and Hang-Seng indices. Nevertheless the interpretation of the parameters of the MGC densities requires a complete study of the moments of every distribution. For example, ρ_{12} does not capture exactly the correlation among both indices, which explains the significant differences of this parameter among the estimated densities. The stationarity conditions of the GARCH processes in all the distributions except the MN are also slightly different from the usual ones as well.[13]

Finally, and for the purpose of illustrating the performance of this family of distributions, the fitted MCGII density with constant variances (Table 1) is depicted in Figure 2. This Figure includes the pictures for the fitted MCGII (left plots: Figures 2.A, 2.C and 2.E) compared to the corresponding fits under the MN (right plots: Figure 2.B, Figure 2.C and Figure 2.F) for different ranges and domains. Particularly, Figures 2.A and 2.B represent the whole domain of the functions, and the rest of the figures illustrate details of the distributions tails. It is noteworthy the fact that the MCGII is capable of capturing different jumps in the probabilistic mass (see Figure 2.C) whilst for the same range the MN density decreases smoothly (see Figure 2.D). Furthermore, the MCGII captures more accurately the leptokurtic density behaviour since it assigns positive probability to areas in the tails where the MN does not (see Figures 2.E and 2.F). This improvement in the accuracy of the MCGII is due to its flexibility to incorporate the whole shape of the density by means of a larger number of parameters than other distributions such as the MN.

These findings can be also illustrated by depicting the marginal densities of every variable computed from the estimates for the multivariate distributions. Figure 3 includes the fitted marginal density for S&P500 under different specifications (MN, MES and MCGII) in comparison to the histogram of the data. Figure 3.A represents the densities for the whole domain whilst Figure 3.B includes only the distributions left tails. From these pictures it is clear that although the MES seems to capture more accurately the sharply peaked density behaviour, the MCGII outperforms the other specifications in the tails. Therefore in the next section we study the out of sample performance of the latter distribution as a representative well-behaved MGC distribution.

[13] See See Ñíguez and Perote (2004) for an example of the GARCH(1,1) stationarity conditions for the particular case of the univariate PES density.

Figure 2: Fitted MCGII density of the S&P500 and the Hang-Seng indices returns for different ranges and domains.

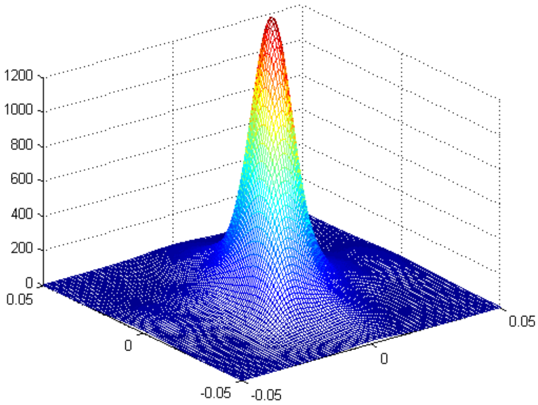


Figure 2.A

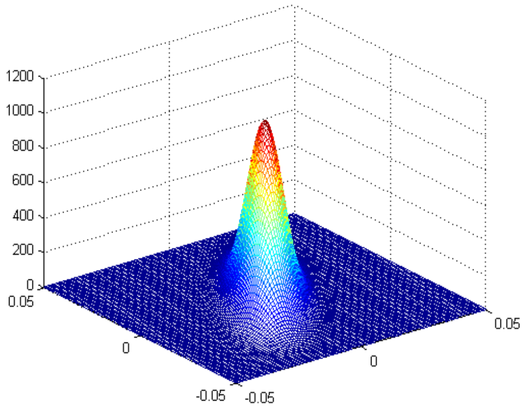


Figure 2.B

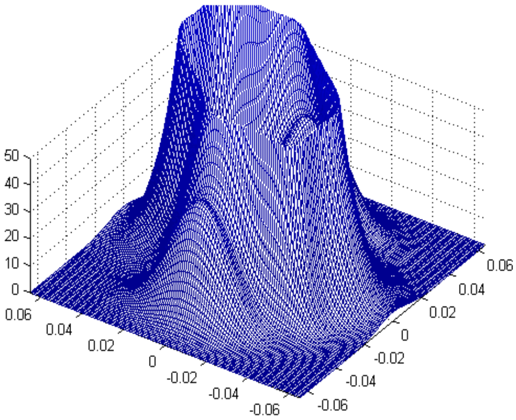


Figure 2.C

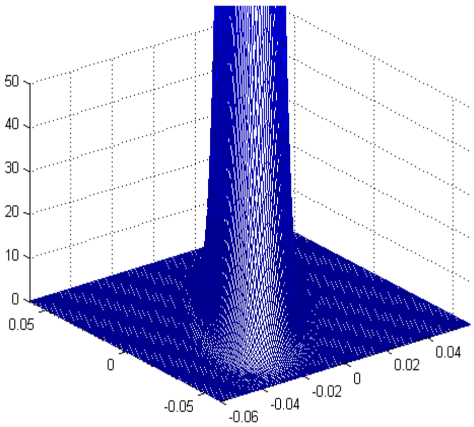


Figure 2.D

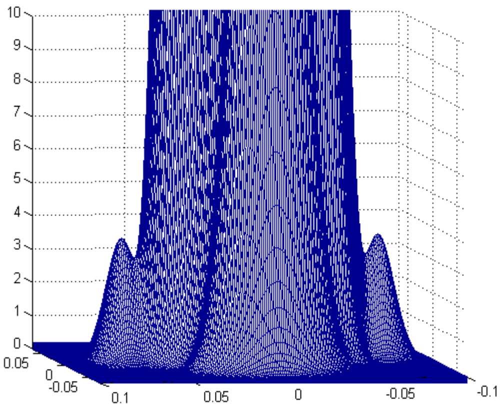


Figure 2.E

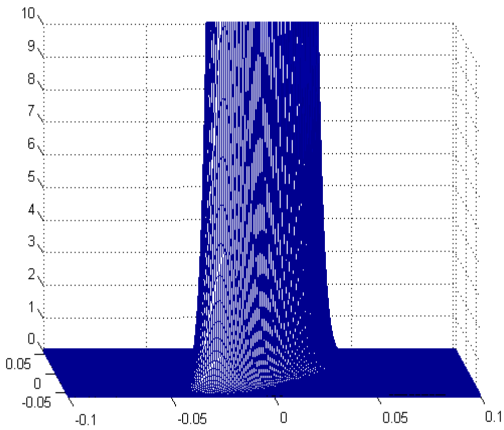
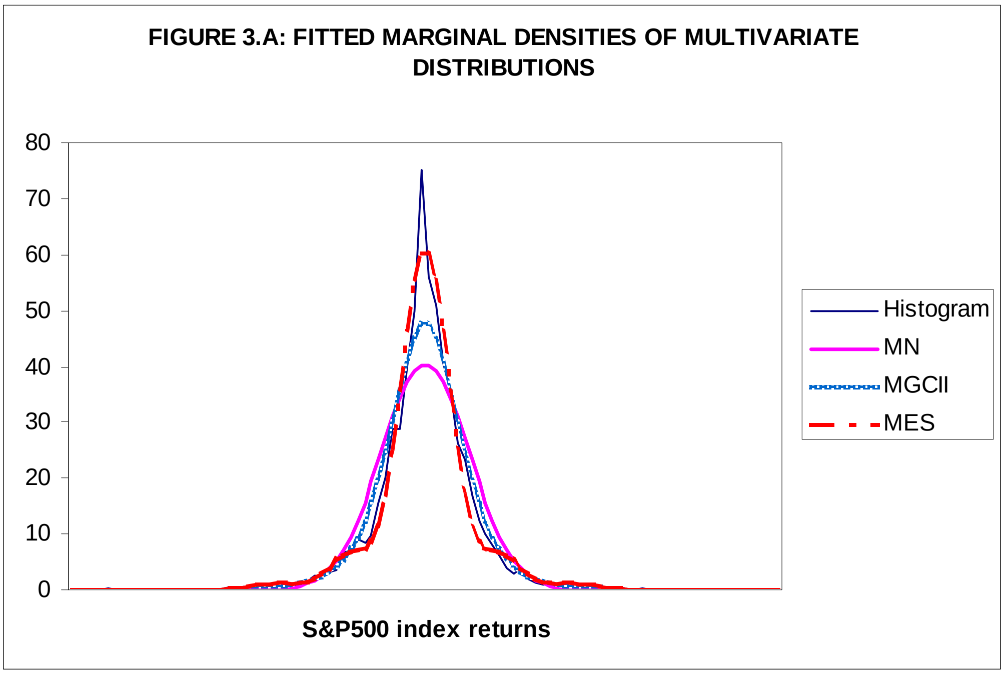
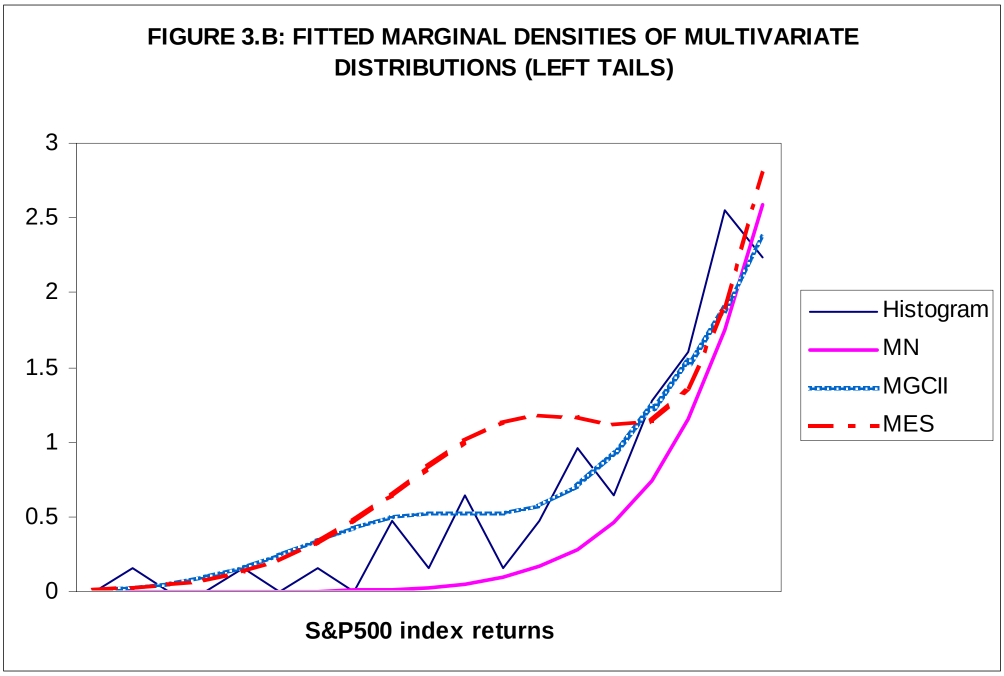


Figure 2.F

Figure 3: Fitted marginal density of S&P500 index computed from the estimated MN, MES and MCGII compared to the histogram of the data.





3.2 Density forecasting

In this section we test the performance of the MCG densities to forecast the full density of the portfolio and compare the forecasts with those of a MN model by using the methodology in Diebold et al. (1998, 1999) and Davidson and MacKinnon (1998). The application of this methodology in a multivariate framework is based on cumulative marginal and conditional distribution functions (cdf), evaluated at the forecasted standardised AR(1) residuals, $\widehat{u}_{it+1} = \frac{\widehat{z}_{it+1}}{\widehat{k}_{it+1}}$ $\forall i = 1, 2$, through the out-of-sample period ($N = 400$ observations). The resulting so-called probability integral transforms (PIT) sequences, labelled $p_{1t},\ p_{2t},\ p_{1|2t},\ p_{2|1t},$, are *i.i.d.* $U(0,1)$ under correct density specification,

$$\begin{aligned} p_{it} &= \int_{-\infty}^{\widehat{u}_{it+1}} f_{it+1}(u_{it+1}) du_{it+1}, \qquad \forall i = 1, 2 \\ p_{i|jt} &= \int_{-\infty}^{\widehat{u}_{it+1}} f_{i|jt+1}(u_{it+1}) du_{it+1} = \frac{\int_{-\infty}^{\widehat{u}_{it+1}} \int_{-\infty}^{\widehat{u}_{jt+1}} f_{t+1}(u_{it+1}, u_{jt+1}) du_{it+1} du_{jt+1}}{\int_{-\infty}^{\widehat{u}_{jt+1}} f_{jt+1}(u_{jt+1}) du_{jt+1}} \qquad \forall i, j = 1, 2 \end{aligned} \tag{13}$$

where $f_{it}(\cdot)$, $f_{i|jt}(\cdot)$ and $f_t(\cdot)$ denote marginal, conditional and joint distributions, respectively.

Moreover since p_{it} is also interpreted as the p-value corresponding to the quantile $\widehat{u}_{it+1}$ of the forecasted density we use the p-value plot methods in Davidson and MacKinnon (1998) to compare the models forecasting performance.[14] So, if the model is correctly specified the difference between the cdf of p_{it} and the 45^0 line should tend to zero asymptotically. The empirical distribution function of p_{it} can be easily computed as,

$$\widehat{P}_{p_{it}}(y_\varrho) = \frac{1}{N} \sum_{t=1}^{\mathbf{N}} \mathbf{1}(p_{it} \leq y_\varrho), \tag{14}$$

where $\mathbf{1}(p_{it} \leq y_\varrho)$ is an indicator function that takes the value 1 if its argument is true and 0 otherwise, and y_ϱ is an arbitrary grid of $\boldsymbol{\varrho}$ points.[15] Alternatively, the p-value discrepancy plot (i.e. plotting $\widehat{P}_{p_{it}}(y_\varrho) - y_\varrho$ against y_ϱ) can be more revealing when it is necessary to discriminate among specifications that perform similarly in terms of the p-value plot (see Fiorentini *et al.,* 2003). Consequently, under correct density specification, the variable $\widehat{P}_{p_{it}}(y_\varrho) - y_\varrho$ must converge to zero.

In Figures 3 and 4 we observe that the MGCII model provides a reasonably good performance for forecasting the full density of the portfolio and clearly overcomes the MN model commonly used in financial applications.

[14] Note that Davidson and MacKinnon (1998) used this method to compare the size and power of hypothesis tests, while following Fiorentini *et al.* (2003) we use it to discriminate among alternative models according to their performance for forecasting the full density.

[15] We use the following $\varrho = 215$ points grid,

$$y_\varrho = 0.001, 0.002, ..., 0.01, 0.015, ..., 0.99, 0.991, ..., 0.999$$

since it highlights the goodness-of-fit in the distribution tails.

Figure 4. P-value plots of the PITs of $\widehat{u}_{it+1}$ obtained under the MCGII and MN models.

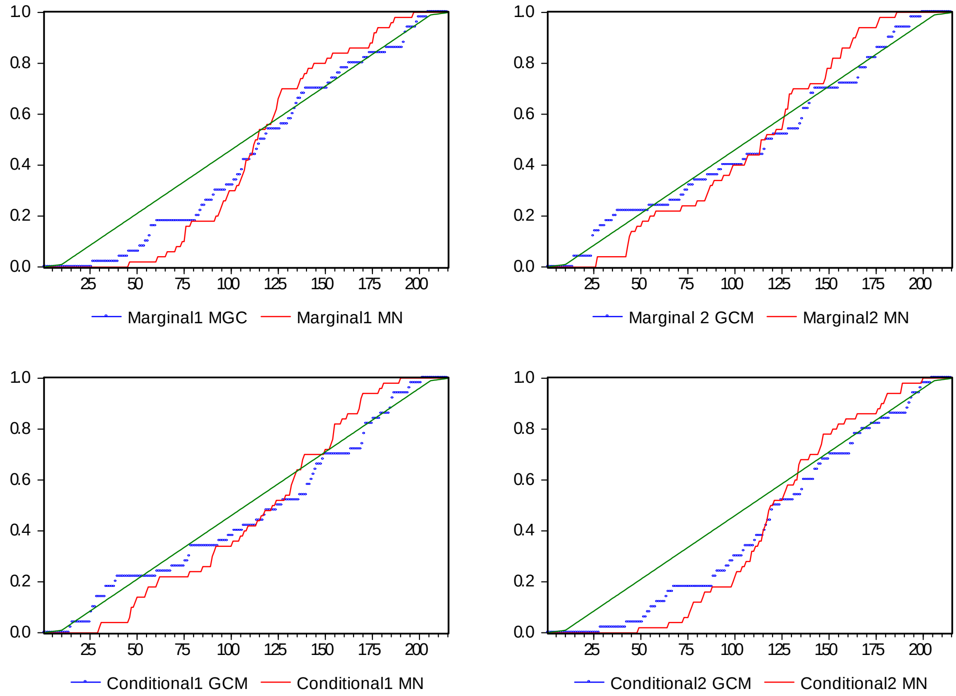
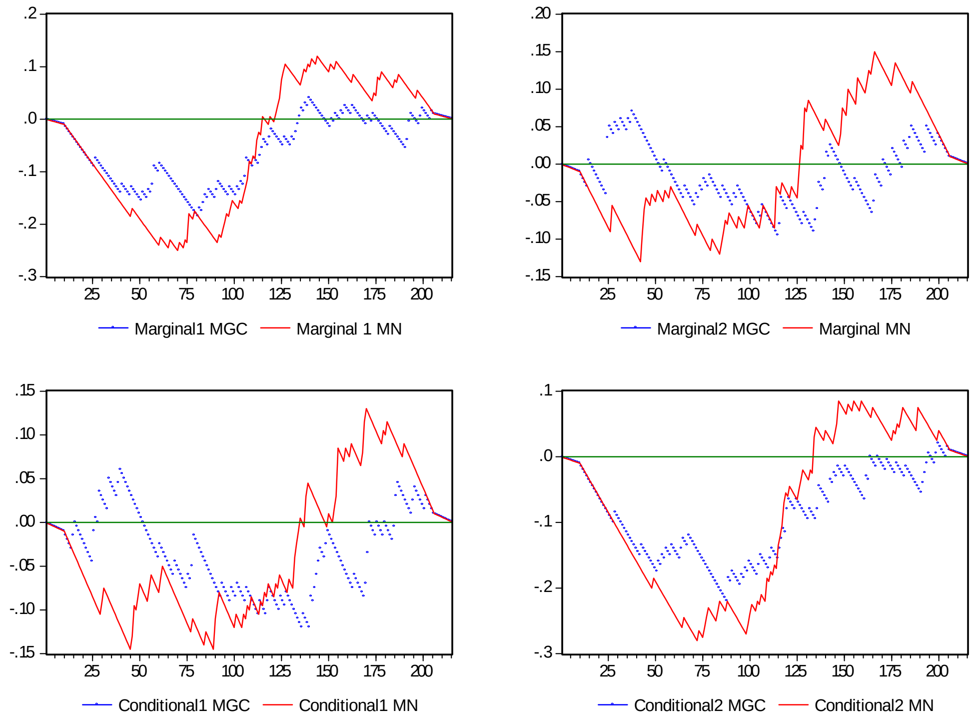


Figure 5: P-value discrepancy plots of the PITs of $\widehat{u}_{it+1}$ obtained under the GCM and MN models.



4 Concluding Remarks

This paper introduces a family of multivariate distributions based on Edgeworth and Gram-Charlier expansions. This family encompasses most of the univariate densities proposed in financial literature (e.g. the so-called Gram-Charlier, Edgeworth-Sargan or Positive Edgeworth-Sargan), which can be obtained as the marginal densities of the different densities nested in this family. Therefore, the MGC densities inherit the properties of their univariate precursories in terms of flexible parameter structure to accurately represent all the characteristic features of most high-frequency financial variables (i.e. thick tails, sharp peak, asymmetries, conditional heteroskedasticity, etc.). Within this family, the specifications that are positive for all the values of the parametric space (and are thus properly defined) merit particular interest. We provide some examples of positive multivariate densities overcoming the deficiencies of the MES density, which can be understood as applications of the Gallant and Nychka (1987) methodology to the multivariate framework. The performance of these densities is compared to fit and forecast the full density of a portfolio of asset returns, and it is found that they perform quite satisfactorily and are superior to the MN, the most commonly used distribution in financial risk management.

Within the multivariate densities based on Edgeworth and Gram-Charlier expansions the MES seems to be more accurate than other positive but more restrictive types of MGC distributions. This evidence highlights the fact that "positive Gram-Charlier expansions" can be adequate representations not only for univariate but also for multivariate densities, but at the cost of a loss of accuracy when compared to other cases (e.g. MES) that do not impose positivity *a priory*. Nevertheless, it must be noted that the MES has several disadvantages compared to MGCI or MGCII including: (*i*) it requires a more careful selection of initial values (or even the implementing of parameter constraints) to avoid the problems caused by possible negative values, (*ii*) the estimates do not guarantee positivity for other estimation techniques further than the maximum likelihood and (*iii*) in some contexts (such as when estimating the density recursively to compute a large number of forecasts) this distribution might give rise to problems of convergence, specifically if the data contain a large number of outliers. Moreover, we show that the MGCII data fits in the tails can be superior than those obtained by the MES. Therefore the choice among the different possibilities within the family depends on other empirical considerations rather than merely on accuracy issues.

This paper opens a hopefully fruitful line of research providing general formulations for MGC densities, and showing evidence of their reasonably good in and out-sample performance through an empirical application. These results suggest that further research seems worthwhile to investigate the model performance for other financial applications, as e.g. asset pricing or credit and market risk forecasting.

Appendix

This appendix includes the proofs of some properties of the MGC densities. Particularly, the constant that makes both the MGCI and the MGCII densities integrate up to one, the marginal densities of MGCI distribution and the cdf for the MGCII are derived. The corresponding proofs for other multivariate densities of the same family can be obtained analogously.

Proof 1:

$$\begin{aligned}
c_i &= \int g(x_i)dx_i + \int \left[\sum_{s=1}^{q} d_{is}H_s(x_i)\right]^2 g(x_i)dx_i + 2\sum_{s=1}^{q} d_{is} \int H_s(x_i)g(x_i)dx_i \\
&= \int g(x_i)dx_i + \int \left[\sum_{s=1}^{q} d_{is}H_s(x_i)\right]^2 g(x_i)dx_i \\
&= 1 + \sum_{s=1}^{q}\sum_{j=1}^{q} d_{is}d_{ij} \int H_s(x_i)H_j(x_i)g(x_i)dx_i \\
&= 1 + \sum_{s=1}^{q} d_{is}^2 \int H_s(x_i)^2 g(x_i)dx_i = 1 + \sum_{s=1}^{q} d_{is}^2 s! \;\blacksquare
\end{aligned}$$

Proof 2:

$$\begin{aligned}
\int \cdots \int F_I(\mathbf{X})dx_1 \cdots dx_n &= \frac{1}{n+1} \int \cdots \int G(\mathbf{X})dx_1 \cdots dx_n + \\
&\frac{1}{n+1} \int \cdots \int \left\{\prod_{i=1}^{n} g\left(x_i\right)\right\} \left\{\sum_{i=1}^{n} \frac{1}{c_i} \left[1 + \sum_{s=1}^{q} d_{is}H_s\left(x_i\right)\right]^2\right\} dx_1 \cdots dx_n \\
&= \frac{1}{n+1} + \frac{n}{n+1} = 1 \;\blacksquare
\end{aligned}$$

Proof 3:

$$\begin{aligned}
f_I(x_i) &= \int \cdots \int F_I(\mathbf{X})dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_n = \frac{1}{n+1} \int \cdots \int G(\mathbf{X})dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_n + \\
&\frac{1}{n+1}\frac{1}{c_i}g(x_i) \left[1 + \sum_{s=1}^{q} d_{is}H_s\left(x_i\right)\right]^2 \int \cdots \int \prod_{j=1,\, j\neq i}^{n} g\left(x_j\right) dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_n + \\
&\frac{1}{n+1}g(x_i) \sum_{j=1,\, j\neq i}^{n} \frac{1}{c_j} \int \cdots \int \left[\prod_{j=1,\, j\neq i}^{n} g\left(x_j\right)\right] \left[1 + \sum_{s=1}^{q} d_{js}H_s\left(x_j\right)\right]^2 dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_n \\
&= \frac{1}{n+1}g(x_i) + \frac{1}{n+1}\frac{1}{c_i}g(x_i) \left[1 + \sum_{s=1}^{q} d_{is}H_s\left(x_i\right)\right]^2 + \frac{n-1}{n+1}g(x_i) \\
&= \frac{n}{n+1}g(x_i) + \frac{1}{n+1}\frac{1}{c_i}g(x_i) \left[1 + \sum_{s=1}^{q} d_{is}H_s\left(x_i\right)\right]^2 \;\blacksquare
\end{aligned}$$

Proof 4:

$$\int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} F_{II}(\mathbf{X}) dx_1 \cdots dx_n$$

$$= \frac{1}{n+1} \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} G(\mathbf{X}) dx_1 \cdots dx_n$$

$$+ \frac{1}{n+1} \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} \left[\prod_{i=1}^{n} g\left(x_i\right) \right] \left\{ \sum_{i=1}^{n} \frac{1}{c_i} \left[1 + \sum_{s=1}^{q} d_{is}^2 H_s\left(x_i\right)^2 \right] \right\} dx_1 \cdots dx_n$$

$$= \frac{1}{n+1} \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} G(\mathbf{X}) dx_1 \cdots dx_n$$

$$+ \frac{1}{n+1} \sum_{i=1}^{n} \int_{-\infty}^{a_i} \frac{1}{c_i} g\left(x_i\right) \left[1 + \sum_{s=1}^{q} d_{is}^2 H_s\left(x_i\right)^2 \right] dx_i \prod_{\substack{j=1 \\ j \neq i}}^{n-1} \int_{-\infty}^{a_j} g\left(x_j\right) dx_j$$

$$= \frac{1}{n+1} \int_{-\infty}^{a_1} \cdots \int_{-\infty}^{a_n} G(\mathbf{X}) dx_1 \cdots dx_n +$$

$$+ \frac{1}{n+1} \sum_{i=1}^{n} \left[\int_{-\infty}^{a_i} g\left(x_i\right) dx_i - \frac{g(a_i)}{c_i} \sum_{s=1}^{q} d_{is}^2 \sum_{k=0}^{s-1} \frac{s!}{(s-k)!} H_{s-k}(a_i) H_{s-k-1}(a_i) \right] \prod_{j=1,\, j \neq i}^{n-1} \int_{-\infty}^{a_j} g\left(x_j\right) dx_j \;\blacksquare$$

See Leon et al (2005) and Ñíguez and Perote (2004) for detailed analyses of the GCMI and GCMII density marginals, respectively.

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